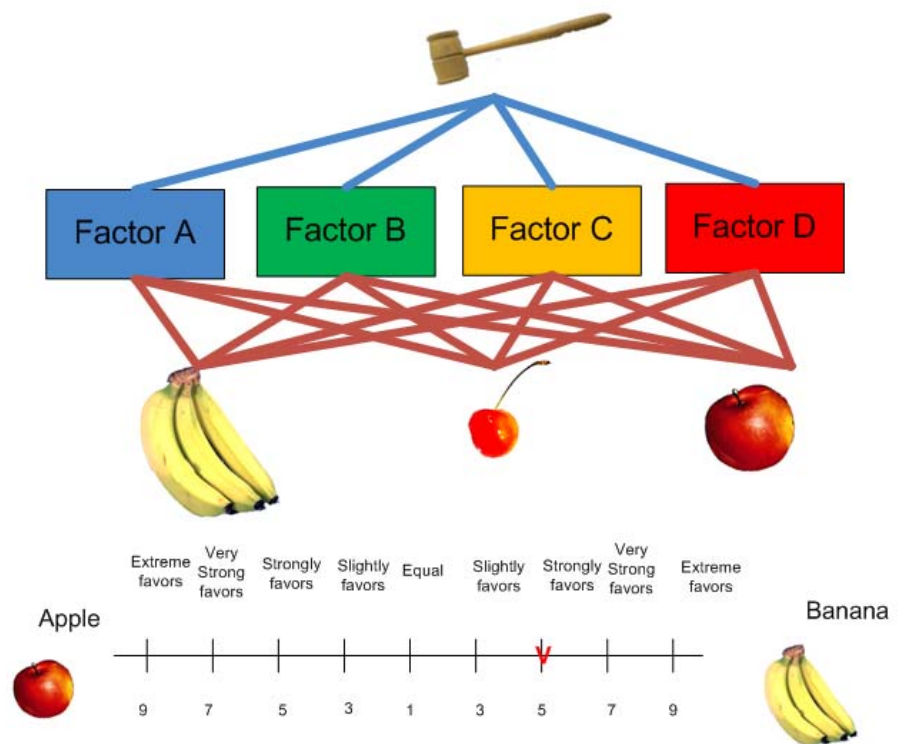


Kardi Teknomo

ANALYTIC HIERARCHY PROCESS (AHP) TUTORIAL



Analytic Hierarchy Process (AHP) Tutorial by Kardi Teknomo

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This tutorial will introduce you to the several methods on multi criteria decision making (MCDM). One famous method of MCDM is called Analytic Hierarchy Process or AHP in short. The AHP procedure had been applied for Decision Support System (DSS), including data mining and machine learning and so many applications. It can involve both subjective human judgments and objective evaluation merely by Eigen vector and examine the consistency of the evaluation by Eigen Value.

Multi Criteria Decision Making

In this section of tutorial, you will learn background materials with several terminologies such as criteria or factors, alternatives choice, evaluation score value and weight of importance level and how to transform different range of judgments into fair evaluation.

Table 1: Example of Goal, Criteria and Alternatives

Goal	Criteria	Alternatives
Decide best school	<ul style="list-style-type: none"> • Distance, • Reputation, • Cost, • Teacher kindliness 	Name of schools under consideration
Finding best apartment	<ul style="list-style-type: none"> • Price, • Down payment, • Distance from shops, • Distance from work/school • Neighbor's Friendliness 	List of apartments under consideration
Select best politician	<ul style="list-style-type: none"> • Charm • Good working program • Benefit for our organization • Attention to our need 	List of candidates
Determine thesis topic	<ul style="list-style-type: none"> • Fast to finish, • Research Cost , • Level of Attractiveness, 	List of thesis topics
Buy car	<ul style="list-style-type: none"> • Initial Price • Operating & Maintenance cost, • Service and comfort, • Status 	Car's trade mark (Honda, GM, Ford, Nissan etc.)
Decide whether to buy or to rent a machine	<ul style="list-style-type: none"> • Total cost (capital, maintenance, operational) • Service • Time to operate • Interconnection with other machines 	Rent or Buy

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We make many decisions in our life. We decide which school to take, which place to live, which clothes to use, which persons to be our best friends or to marry, which food to eat, which car to buy, and so on.

Decision making is process to choose among alternatives based on multiple criteria. In each of these decisions, deep in our mind we have several factors or **criteria** on what to consider and we also have several **alternatives** choices that we should decide. On group decision making these criteria and alternatives are more obvious and must be determined first before we give some judgment score or evaluation values on them. In this tutorial, I will use the word 'factors' and 'criteria' interchangeably. Similarly, I use 'alternative' and 'choice' for the same meaning. Table 1 shows example of criteria and alternatives of several decision makings.

The determination of criteria and alternatives are very subjective. Notice that the list of criteria and alternatives above are not exhausted list. They neither cover all possible criteria nor all possible alternatives. There is no correct or wrong criterion because it is subjective opinion. Different people may add or subtract those lists. Some factors may be combined together and some criterion may be broken down into more detail criteria.

Most of decisions makings are based on individual judgments. As we try to make our decision as rational as possible we need to quantify these subjective opinions into subjective **values**. The values are number within any certain range; say from 1 to 10 or -5 to 5. The values can be any number with order (ordinal number) and you can even put different range for each factor. Higher value indicates higher level of the factor or preferable values. Now you see that not only the criteria and alternatives are subjective, even the values are also subjective. They are depending on you as decision maker.

Cross Tabulation

The simplest multi criteria decision making is to put into a cross table of criteria and alternatives. Then we put subjective score value on each cell of the table. The sum (or normalized sum) of and compute the sum of all factors for each alternatives.

Table 2: Evaluation based on scores of each factor

Criteria Alternatives	Choice X	Choice Y	Choice Z	Range
Factor A	1	4	5	0-5
Factor B	20	70	50	1-100
Factor C	-2	0	1	-2 to +2
Factor D	0.4	0.75	0.4	0 to 1
Sum	19.4	74.75	56.4	
Normalized Score	12.9%	49.7%	37.5%	

For example, we have 3 alternative choices X, Y and Z and four criteria to decide the alternatives A, B, C and D. You can input any name for alternatives and criteria. The values on the table 2 are any

number certain range for each factor. The only similarity between these numbers is that they have the same interpretation that higher values are preferable than smaller values.

If you have many alternatives, sometimes it is easier to compare the sum value of each choice by normalizing them. Total sums is 150.55 (=19.4+74.75+56.4). The sum of each choice is normalized by division of each sum with the total sums. For instance, choice X is normalized into $19.4/150.55 \times 100\% = 12.9\%$. Clearly choice Y is preferable than choice Z while choice Z is better than X.

However, you will notice that the range of value for each factors are not the same. It is quite unfair to sum all the values of multiple criteria and compare the result. Clearly factor B is dominant because the range has higher value. To be fair, we can propose two solutions:

1. Instead of using arbitrary values for each factor, we just rank the choice for each factor. Smaller rank value is more preferable than higher rank.
2. We transform the score value of each factor according to the range value such that each factor will have the same range.

In the next sections, let us try the two solutions one by one.

Evaluation based on Rank

Now we change the value of table 2 above into rank.

Table 3: Evaluation based on ranks of each factor

Criteria Alternatives	Choice X	Choice Y	Choice Z
Factor A	3	2	1
Factor B	3	1	2
Factor C	3	2	1
Factor D	2	1	2
Sum	11	6	6
Normalized Score	26.09%	36.96%	36.96%

The values of each row are either 1 or 2 or 3 represent the rank (based on the value of previous table). Since smaller rank value is more preferable than higher rank, we need to normalize the sum in different way using formula below

$$normalized\ score = \frac{1}{2} \left(1 - \frac{sum}{total\ sum} \right)$$

The total sum is 23 (=11+6+6). In this case the normalized score of Choice X is $0.5 \times (1 - 11/23) = 26.09\%$, while the normalized score of Choice Y and Z are $0.5 \times (1 - 6/23) = 36.96\%$. In this case higher normalized score correspond to higher preference. You may notice that we have

transformed the rank values (which is ordinal scale) into normalized score value (which is a ratio scale).

Comparing the results of two tables above show that the rank of preference change by the way we compute our case. Even though we based our judgments on the same score values, the rank reduce some information of these values. In this case choice Y and Z become indifference, or equally preferable.

Now let us see what happen if we transform the score value of each factor in such a way such that all factors have the same range value. Say, we choose all factors to have range to be 0 to 1. To convert linearly the score of each factor from table 2 into table 4, we use the following formula which is based on simple geometric of a line segment

$$new\ score = \frac{nub - nlb}{oub - olb} (original\ score - olb) + nlb$$

The geometry of the linear transformation is shown in the figure below

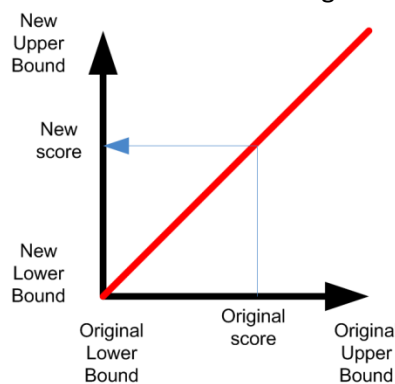


Table 4: Converted New Scores based on Range

Criteria Alternatives	Choice X	Choice Y	Choice Z
Factor A	0.2	0.8	1
Factor B	0.192	0.697	0.495
Factor C	0	0.5	0.75
Factor D	0.4	0.75	0.4
Sum	0.792	2.747	2.645
Normalized Score	12.8%	44.4%	42.8%

For instance, Factor A has originally range 0 to 5. To make score of choice Y from 4 into a range of 0 to 1 we have $olb = 0$, $oub = 5$, $nlb = 0$, $nub = 1$, and $score = 4$, thus

$$new\ score = \frac{1-0}{5-0} (4-0) + 0 = \frac{4}{5} = 0.8. \text{ Another example, for choice X in factor B has original}$$

score of 20 and original range 1 to 100. Thus we have $o_{lb} = 1$, $o_{ub} = 100$, $n_{lb} = 0$, $n_{ub} = 1$ and score = 20, thus $new\ score = \frac{1-0}{100-1}(20-1)+0 = \frac{19}{99} = 0.192$

Clearly the transformation of score value is a little bit more complicated than rank but we get better results.

In the next section you will learn more general method.

Weighted Criteria

Having a fair decision table as shown in Table 4, now come out another question. What happen if the factors have different importance weight? Of course the weight of importance is subjective value, but we would like to know how the result will change if we put different weight on each factor.

Just for example we judge that factor B and C are 2 times more important than factor D while factor A is 3 times more important than factor B. We normalized the subjective judgment of importance level and we obtain weight of importance as shown in Table 5

Table 5: Weight of Importance

	Factor A	Factor B	Factor C	Factor D	Sum
Importance Level	6	2	2	1	11
Importance Weight	54.5%	18.2%	18.2%	9.1%	100.0%

Having the normalized weight of each factor, now we can multiply the converted score of table 4 with the normalized weight and get the new weighted score as show in table 6.

Table 6: Weighted scores

Criteria Alternatives	Weight	Choice X	Choice Y	Choice Z
Factor A	54.5%	0.109	0.436	0.545
Factor B	18.2%	0.035	0.127	0.090
Factor C	18.2%	0.000	0.091	0.136
Factor D	9.1%	0.036	0.068	0.036
Sum	100.0%	0.180	0.722	0.808
Normalized Score		10.5%	42.2%	47.2%

Comparing the normalized score of Table 4 and Table 6 we can observed some shift on the choice. In Table 4, choice Y is preferable than Z. However, after we include the weight of importance of each factor, we conclude that choice Z is the most preferable alternative.

Rank Reversal

In this section, I will show that rank aggregation will lead to rank reversal compared to score aggregation.

Suppose 5 judges have to evaluate 10 types of items. Each judge gives score 1 to 100 for each item. Here is an example of their judgments.

Score	Judge A	Judge B	Judge C	Judge D	Judge E
item 1	70	80	80	50	100
item 2	60	90	70	60	60
item 3	70	90	80	80	90
item 4	80	60	70	50	90
item 5	60	90	50	90	60
item 6	100	60	70	60	70
item 7	80	70	90	60	60
item 8	80	80	50	70	50
item 9	50	100	80	60	100
item 10	60	100	80	90	90

Since all judges are considered equally experts, their weights are equal. Thus, we can either sum their scores or take average of their scores. Our goal in evaluating the items is to rank the items. Table below show the aggregation results and we rank the average (or the sum) of the scores.

Score	Judge A	Judge B	Judge C	Judge D	Judge E	Aggregation		
						sum	average	rank
item 1	70	80	80	50	100	380	76	4
item 2	60	90	70	60	60	340	68	9
item 3	70	90	80	80	90	410	82	2
item 4	80	60	70	50	90	350	70	7
item 5	60	90	50	90	60	350	70	7
item 6	100	60	70	60	70	360	72	5
item 7	80	70	90	60	60	360	72	5
item 8	80	80	50	70	50	330	66	10
item 9	50	100	80	60	100	390	78	3
item 10	60	100	80	90	90	420	84	1

Now suppose we have another scenario that the judges want to use their rank instead of their scores. In this case, each judge will rank their scores. Here are their ranks based on the scores above.

AHP Tutorial

Rank	Judge A	Judge B	Judge C	Judge D	Judge E
item 1	5	6	2	9	1
item 2	7	3	6	5	7
item 3	5	3	2	3	3
item 4	2	9	6	9	3
item 5	7	3	9	1	7
item 6	1	9	6	5	6
item 7	2	8	1	5	7
item 8	2	6	9	4	10
item 9	10	1	2	5	1
item 10	7	1	2	1	3

To aggregate the rank, they use the same way as aggregating the scores that is using sum or average. However, this time, we aggregate the ranks instead of the scores. Then, they sort the rank aggregation using based on minimum rank aggregation.

Rank	Judge A	Judge B	Judge C	Judge D	Judge E	Aggregation		
						sum	average	rank
item 1	5	6	2	9	1	23	4.6	4
item 2	7	3	6	5	7	28	5.6	8
item 3	5	3	2	3	3	16	3.2	2
item 4	2	9	6	9	3	29	5.8	9
item 5	7	3	9	1	7	27	5.4	6
item 6	1	9	6	5	6	27	5.4	6
item 7	2	8	1	5	7	23	4.6	4
item 8	2	6	9	4	10	31	6.2	10
item 9	10	1	2	5	1	19	3.8	3
item 10	7	1	2	1	3	14	2.8	1

Notice that the rank based on the aggregation of scores is not the same as the rank based on the aggregation of rank. Some item will have reverse order. That is what we called as *rank reversal*. In general, rank reversal is the rule of rank aggregation. The similarity between rank of scores and rank of rank is just incidental. In the example above, item 2 suppose to have rank 9 but using rank aggregation, it becomes rank 8. On the other hand, item 4 suppose to have rank 7 but using rank aggregation, now it goes down to rank 9.

	Rank of		
	score	rank	
item 1	4	4	●
item 2	9	8	●
item 3	2	2	●
item 4	7	9	●
item 5	7	6	●
item 6	5	6	●
item 7	5	4	●
item 8	10	10	●
item 9	3	3	●
item 10	1	1	●

The simple lesson is: use score aggregation, and not rank aggregation because it may lead to rank reversal.

We have learned simple method to quantify our subjective opinion for our decision making. In the next section you will learn another powerful method called Analytic Hierarchy Process (AHP).

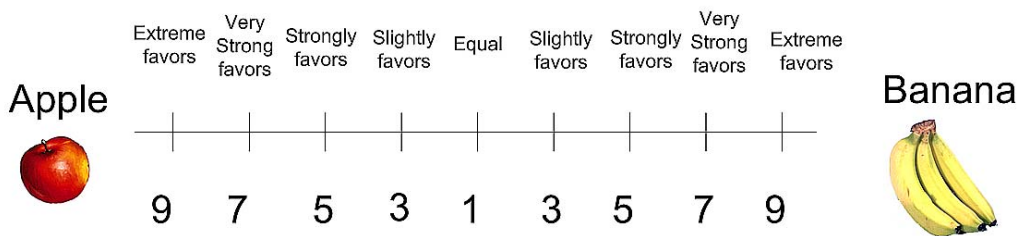
Analytic Hierarchy Process (AHP)

Analytic Hierarchy Process (AHP) is one of Multi Criteria decision making method that was originally developed by Prof. Thomas L. Saaty. In short, it is a method to derive ratio scales from paired comparisons. The input can be obtained from actual measurement such as price, weight etc., or from subjective opinion such as satisfaction feelings and preference. AHP allow some small inconsistency in judgment because human is not always consistent. The ratio scales are derived from the principal Eigen vectors and the consistency index is derived from the principal Eigen value.

Don't worry if you don't understand yet about all of those terminologies above because the purpose of this tutorial is to explain that in a very simple way. You just need to read on and at the end you will understand. This tutorial is also accompanied with a MS Excel file (yes you can do AHP with spreadsheet).

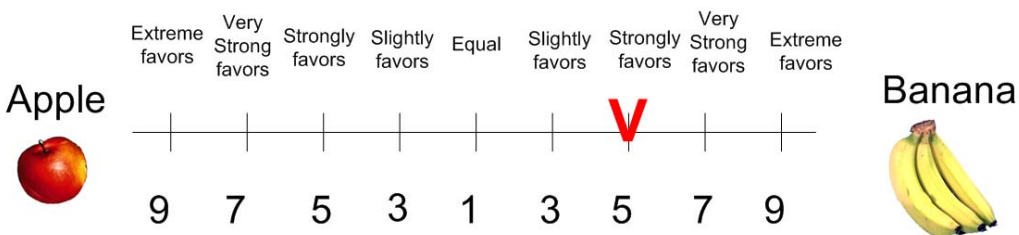
Pair-wise Comparison

Now let me explain what paired comparison is. It is always easier to explain by an example. Suppose we have two fruits **Apple** and **Banana**. I would like to ask you, which fruit you like better than the other and how much you like it *in comparison* with the other. Let us make a relative scale to measure how much you like the fruit on the left (Apple) compared to the fruit on the right (Banana).

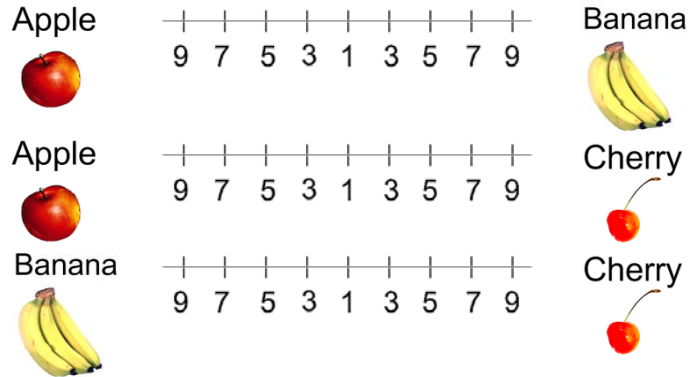


If you like the apple better than banana, you thick a mark between number 1 and 9 on left side, while if you favor banana more than apple, then you mark on the right side.

For instance I strongly favor banana to apple then I give mark like this



Now suppose you have three choices of fruits. Then the pair wise comparison goes as the following



You may observe that the number of comparisons is a combination of the number of things to be compared. Since we have 3 objects (Apple, Banana and Cheery), we have 3 comparisons. Table below shows the number of comparisons.

Table 7: Number of comparisons

Number of things	1	2	3	4	5	6	7	n
number of comparisons	0	1	3	6	10	15	21	$\frac{n(n-1)}{2}$

The scaling is not necessary 1 to 9 but for qualitative data such as preference, ranking and subjective opinions, it is suggested to use scale 1 to 9.

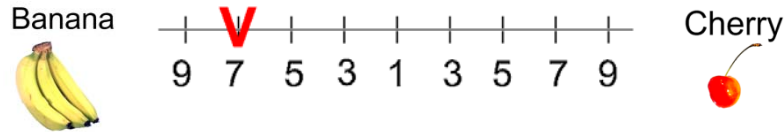
In the next section, you will learn how to analyze these paired comparisons

Making Comparison Matrix

By now you know how to make paired comparisons. In this section you will learn how to make a reciprocal matrix from pair wise comparisons.

For example John has 3 kinds of fruits to be compared and he made subjective judgment on which fruit he likes best, like the following





We can make a matrix from the 3 comparisons above. Because we have three comparisons, thus we have 3 by 3 matrix. The diagonal elements of the matrix are always 1 and we only need to fill up the upper triangular matrix. How to fill up the upper triangular matrix is using the following rules:

1. If the judgment value is on the **left** side of 1, we put the **actual judgment** value.
2. If the judgment value is on the **right** side of 1, we put the **reciprocal** value.

Comparing apple and banana, John slightly favor banana, thus we put $\frac{1}{3}$ in the row 1 column 2 of the matrix. Comparing Apple and Cherry, John strongly likes apple, thus we put actual judgment 5 on the first row, last column of the matrix. Comparing banana and cherry, banana is dominant. Thus we put his actual judgment on the second row, last column of the matrix. Then based on his preference values above, we have a reciprocal matrix like this

$$\mathbf{A} = \begin{matrix} & \begin{matrix} apple & banana & cerry \end{matrix} \\ \begin{matrix} apple \\ banana \\ cerry \end{matrix} & \begin{bmatrix} 1 & \frac{1}{3} & 5 \\ & 1 & 7 \\ & & 1 \end{bmatrix} \end{matrix}$$

To fill the lower triangular matrix, we use the reciprocal values of the upper diagonal. If a_{ij} is the element of row i column j of the matrix, then the lower diagonal is filled using this formula

$$a_{ji} = \frac{1}{a_{ij}}$$

Thus now we have complete comparison matrix

$$\mathbf{A} = \begin{matrix} & \begin{matrix} apple & banana & cerry \end{matrix} \\ \begin{matrix} apple \\ banana \\ cerry \end{matrix} & \begin{bmatrix} 1 & \frac{1}{3} & 5 \\ 3 & 1 & 7 \\ \frac{1}{5} & \frac{1}{7} & 1 \end{bmatrix} \end{matrix}$$

Notice that all the element in the comparison matrix are positive, or $a_{ij} > 0$. Next section will discuss about how you will use this matrix.

How to compute Eigen Value and Eigen vector?

In this section, I would like to explain what the meaning of Eigen vector and Eigen value is and how to compute them manually.

Priority Vectors

Having a comparison matrix, now we would like to compute priority vector, which is the normalized Eigen vector of the matrix. If you would like to know what the meaning of Eigen vector and Eigen value is and how to compute them manually, go to my other tutorial and then return back here. The method that I am going to explain in this section is only an approximation of Eigen vector (and Eigen value) of a reciprocal matrix. This approximation is actually worked well for small matrix size $n \leq 3$ and there is no guarantee that the rank will not reverse because of the approximation error. Nevertheless it is easy to compute because all we need to do is just to normalize each column of the matrix. At the end I will show the error of this approximation.

Suppose we have 3 by 3 reciprocal matrix from paired comparison

$$\mathbf{A} = \begin{array}{c} \text{apple} \\ \text{banana} \\ \text{cerry} \end{array} \begin{array}{ccc} \text{apple} & \text{banana} & \text{cerry} \\ \left[\begin{array}{ccc} 1 & \frac{1}{3} & 5 \\ 3 & 1 & 7 \\ \frac{1}{5} & \frac{1}{7} & 1 \end{array} \right] \end{array}$$

We sum each column of the reciprocal matrix to get

$$\mathbf{A} = \begin{array}{c} \text{apple} \\ \text{banana} \\ \text{cerry} \\ \text{sum} \end{array} \begin{array}{ccc} \text{apple} & \text{banana} & \text{cerry} \\ \left[\begin{array}{ccc} 1 & \frac{1}{3} & 5 \\ 3 & 1 & 7 \\ \frac{1}{5} & \frac{1}{7} & 1 \\ \frac{21}{5} & \frac{31}{21} & 13 \end{array} \right] \end{array}$$

Then we divide each element of the matrix with the sum of its column, we have normalized relative weight. The sum of each column is 1.

$$\mathbf{A} = \begin{array}{c} \text{apple} \\ \text{banana} \\ \text{cerry} \\ \text{sum} \end{array} \begin{array}{ccc} \text{apple} & \text{banana} & \text{cerry} \\ \left[\begin{array}{ccc} \frac{5}{21} & \frac{7}{31} & \frac{5}{13} \\ \frac{15}{21} & \frac{21}{31} & \frac{7}{13} \\ \frac{1}{21} & \frac{3}{31} & \frac{1}{13} \\ 1 & 1 & 1 \end{array} \right] \end{array}$$

The normalized principal Eigen vector can be obtained by averaging across the rows

$$\mathbf{w} = \frac{1}{3} \begin{bmatrix} \frac{5}{21} + \frac{7}{31} + \frac{5}{13} \\ \frac{15}{21} + \frac{21}{31} + \frac{7}{13} \\ \frac{1}{21} + \frac{3}{31} + \frac{1}{13} \end{bmatrix} = \begin{bmatrix} 0.2828 \\ 0.6434 \\ 0.0738 \end{bmatrix}$$

The normalized principal Eigen vector is also called **priority vector**. Since it is normalized, the sum of all elements in priority vector is 1. The priority vector shows relative weights among the things that we compare. In our example above, Apple is 28.28%, Banana is 64.34% and Cherry is 7.38%. John most preferable fruit is Banana, followed by Apple and Cheery. In this case, we know more than their ranking. In fact, the relative weight is a ratio scale that we can divide among them. For example, we can say that John likes banana 2.27 (=64.34/28.28) times more than apple and he also like banana so much 8.72 (=64.34/7.38) times more than cheery.

Aside from the relative weight, we can also check the consistency of John's answer. To do that, we need what is called Principal Eigen value. Principal Eigen value is obtained from the summation of products between each element of Eigen vector and the sum of columns of the reciprocal matrix.

$$\lambda_{\max} = \frac{21}{5} (0.2828) + \frac{31}{21} (0.6434) + 13(0.0738) = 3.0967$$

Computation and the meaning of consistency are explained in the next section.

As a note, I put the comparison matrix into Maxima software to see how different is the result of numerical computation of Eigen value and Eigen vector compared to the approximation above.

$$\mathbf{A} = \begin{bmatrix} 1 & \frac{1}{3} & 5 \\ 3 & 1 & 7 \\ \frac{1}{5} & \frac{1}{7} & 1 \end{bmatrix}$$

$$[\mathbf{W}, \boldsymbol{\lambda}] = \text{eig}(\mathbf{A})$$

We get three Eigen vectors concatenated into 3 columns of matrix \mathbf{W}

$$\mathbf{W} = \begin{bmatrix} 0.3928 & -0.1964 + 0.3402i & -0.1964 - 0.3402i \\ 0.9140 & 0.9140 & 0.9140 \\ 0.1013 & -0.0506 - 0.0877i & -0.0506 + 0.0877i \end{bmatrix}$$

The corresponding Eigen values are the diagonal of matrix $\boldsymbol{\lambda}$

$$\boldsymbol{\lambda} = \begin{bmatrix} 3.0649 & 0 & 0 \\ 0 & -0.0324 + 0.4448i & 0 \\ 0 & 0 & -0.0324 - 0.4448i \end{bmatrix}$$

The largest Eigen value is called the Principal Eigen value, that is $\lambda_{\max}^* = 3.0649$ which is very close

to our approximation $\lambda_{\max} = 3.0967$ (about 1% error). The principal Eigen vector is the Eigen vector that corresponds to the highest Eigen value.

$$\bar{\mathbf{w}} = \begin{bmatrix} 0.3928 \\ 0.9140 \\ 0.1013 \end{bmatrix}$$

The sum is 1.4081 and the normalized principal Eigen vector is

$$\mathbf{w}^* = \begin{bmatrix} 0.2790 \\ 0.6491 \\ 0.0719 \end{bmatrix}$$

This result is also very close to our approximation

$$\mathbf{w} = \begin{bmatrix} 0.2828 \\ 0.6434 \\ 0.0738 \end{bmatrix}$$

Thus the approximation is quite good.

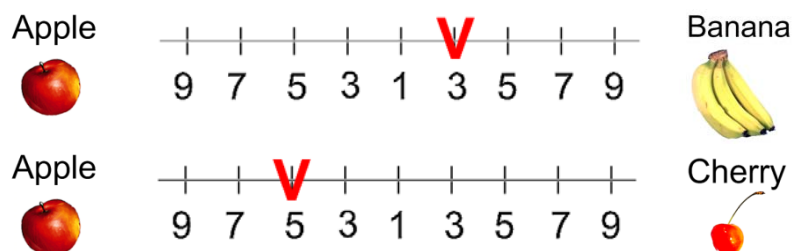
Thus the sum of Eigen vector is not one. When you normalized an Eigen vector, then you get a priority vector. The sum of priority vector is one.

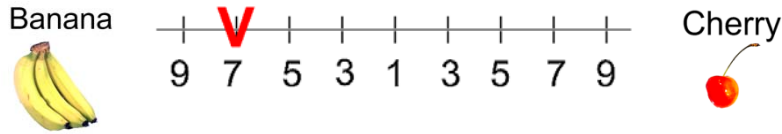
In the next section you will learn how to make use of information of Principal Eigen value to measure whether the opinion is consistent.

Consistency Index and Consistency Ratio

What is the meaning that our opinion is consistent? How do we measure the consistency of subjective judgment? At the end of this section will be able to answer those questions.

Let us look again on John's judgment that we discussed in the previous section. Is John judgment consistent or not?





First he prefers Banana to Apple. Thus we say that for John, Banana has greater value than Apple. We write it as $B \succ A$.

Next, he prefers Apple to Cherry. For him, Apple has greater value than Cherry. We write it as $A \succ C$.

Since $B \succ A$ and $A \succ C$, logically, we hope that $B \succ C$ or Banana must be preferable than Cherry. This logic of preference is called *transitive* property. If John answers in the last comparison is transitive (that he like Banana more than Cherry), then his judgment is consistent. On the contrary, if John prefers Cherry to Banana then his answer is inconsistent. Thus consistency is closely related to the transitive property.

A comparison matrix A is said to be consistent if $a_{ij} a_{jk} = a_{ik}$ for all i, j and k . However, we shall not force the consistency. For example, $B \succ A$ has value $3 \succ 1$ and $A \succ C$ has value $5 \succ 1$, we shall not insist that $B \succ C$ must have value $15 \succ 1$. This too much consistency is undesirable because we are dealing with human judgment. To be called *consistent*, the rank can be transitive but the values of judgment are not necessarily forced to multiplication formula $a_{ij} a_{jk} = a_{ik}$.

Prof. Saaty proved that for consistent reciprocal matrix, the largest Eigen value is equal to the size of comparison matrix, or $\lambda_{\max} = n$. Then he gave a measure of consistency, called Consistency Index as deviation or degree of consistency using the following formula

$$CI = \frac{\lambda_{\max} - n}{n - 1}$$

Thus in our previous example, we have $\lambda_{\max} = 3.0967$ and the size of comparison matrix is $n = 3$, thus the consistency index is

$$CI = \frac{\lambda_{\max} - n}{n - 1} = \frac{3.0967 - 3}{2} = 0.0484$$

Knowing the Consistency Index, the next question is how do we use this index? Again, Prof. Saaty proposed that we use this index by comparing it with the appropriate one. The appropriate Consistency index is called Random Consistency Index (RI).

He randomly generated reciprocal matrix using scale $\frac{1}{9}, \frac{1}{8}, \dots, 1, \dots, 8, 9$ (similar to the idea of Bootstrap) and get the random consistency index to see if it is about 10% or less. The average random consistency index of sample size 500 matrices is shown in the table below

Table 8: Random Consistency Index (*RI*)

n	1	2	3	4	5	6	7	8	9	10
RI	0	0	0.58	0.9	1.12	1.24	1.32	1.41	1.45	1.49

Then, he proposed what is called Consistency Ratio, which is a comparison between Consistency Index and Random Consistency Index, or in formula

$$CR = \frac{CI}{RI}$$

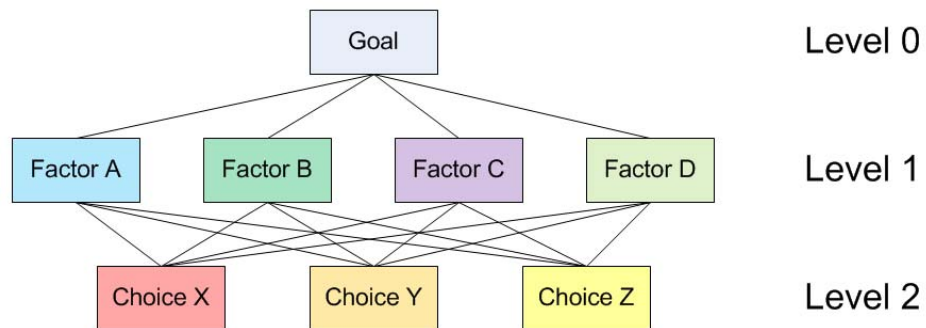
If the value of Consistency Ratio is smaller or equal to 10%, the inconsistency is acceptable. If the Consistency Ratio is greater than 10%, we need to revise the subjective judgment.

For our previous example, we have $CI = 0.0484$ and RI for $n = 3$ is 0.58, then we have $CR = \frac{CI}{RI} = \frac{0.0484}{0.58} = 8.3\% < 10\%$. Thus, John's subjective evaluation about his fruit preference is consistent.

So far, in AHP we are only dealing with paired comparison of criteria or alternative but not both. In next section, I show an example to use both criteria and alternative in two levels of AHP.

Illustrative Example of Analytic Hierarchy Process

In this section, I show an example of two levels AHP. The structure of hierarchy in this example can be drawn as the following



Level 0 is the goal of the analysis. Level 1 is multi criteria that consist of several factors. You can also add several other levels of sub criteria and sub-sub criteria but I did not use that here. The last level (level 2 in figure above) is the alternative choices. You can see again Table 1 for several examples of Goals, factors and alternative choices. The lines between levels indicate relationship between factors, choices and goal. In level 1 you will have one comparison matrix corresponds to pair-wise comparisons between 4 factors with respect to the goal. Thus, the comparison matrix of

level 1 has size of 4 by 4. Because each choice is connected to each factor, and you have 3 choices and 4 factors, then in general you will have 4 comparison matrices at level 2. Each of these matrices has size 3 by 3. However, in this particular example, you will see that some weight of level 2 matrices are too small to contribute to overall decision, thus we can ignore them.

Based on questionnaire survey or your own paired comparison, we make several comparison matrices. Click here if you do not remember how to make a comparison matrix from paired comparisons. Suppose we have comparison matrix at level 1 as table below. The yellow color cells in upper triangular matrix indicate the parts that you can change in the spreadsheet. The diagonal is always 1 and the lower triangular matrix is filled using formula $a_{ji} = \frac{1}{a_{ij}}$.

Table 9: Paired comparison matrix level 1 with respect to the goal

Criteria	A	B	C	D	Priority Vector
A	1.00	3.00	7.00	9.00	57.39%
B	0.33	1.00	5.00	7.00	29.13%
C	0.14	0.20	1.00	3.00	9.03%
D	0.11	0.14	0.33	1.00	4.45%
Sum	1.59	4.34	13.33	20.00	100.00%

$$\lambda_{\max} = 4.2692, CI = 0.0897, CR = 9.97\% < 10\% \text{ (acceptable)}$$

The priority vector is obtained from normalized Eigen vector of the matrix. Click here if you do not remember how to compute priority vector and largest Eigen value λ_{\max} from a comparison matrix. CI and CR are consistency Index and Consistency ratio respectively, as I have explained in previous section. For your clarity, I include again here some part of the computation:

$$\lambda_{\max} = (0.5739)(1.59) + (0.2913)(4.34) + (0.0903)(13.33) + (0.0445)(20) = 4.2692$$

$$CI = \frac{\lambda_{\max} - n}{n - 1} = \frac{4.2692 - 4}{3} = 0.0897$$

$$CR = \frac{CI}{RI} = \frac{0.0897}{0.90} = 9.97\% < 10\% \text{ (Thus, OK because quite consistent)}$$

Random Consistency Index (RI) is obtained from Table 8.

Suppose you also have several comparison matrices at level 2. These comparison matrices are made for each choice, with respect to each factor.

Table 10: Paired comparison matrix level 2 with respect to Factor A

Choice	X	Y	Z	Priority Vector
X	1.00	1.00	7.00	51.05%
Y	1.00	1.00	3.00	38.93%
Z	0.14	0.33	1.00	10.01%
Sum	2.14	2.33	11.00	100.00%

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$$\lambda_{\max} = 3.104, \text{ CI} = 0.05, \text{ CR} = 8.97\% < 10\% \text{ (acceptable)}$$

Table 11: Paired comparison matrix level 2 with respect to Factor B

Choice	X	Y	Z	Priority Vector
X	1.00	0.20	0.50	11.49%
Y	5.00	1.00	5.00	70.28%
Z	2.00	0.20	1.00	18.22%
Sum	8.00	1.40	6.50	100.00%

$$\lambda_{\max} = 3.088, \text{ CI} = 0.04, \text{ CR} = 7.58\% < 10\% \text{ (acceptable)}$$

We can do the same for paired comparison with respect to Factor C and D. However, the weight of factor C and D are very small (look at Table 9 again, they are only about 9% and 5% respectively), therefore we can assume the effect of leaving them out from further consideration is negligible. We ignore these two weights as set them as zero. So we do not use the paired comparison matrix level 2 with respect to Factor C and D. In that case, the weight of factor A and B in Table 9 must be adjusted so that the sum still 100%

$$\text{Adjusted weight for factor A} = \frac{57.39\%}{57.39\% + 29.13\%} = 0.663$$

$$\text{Adjusted weight for factor B} = \frac{29.13\%}{57.39\% + 29.13\%} = 0.337$$

Then we compute the overall composite weight of each alternative choice based on the weight of level 1 and level 2. The overall weight is just normalization of linear combination of multiplication between weight and priority vector.

$$X = (0.663)(51.05\%) + (0.337)(11.49\%) = 37.72\%$$

$$Y = (0.663)(38.93\%) + (0.337)(70.28\%) = 49.49\%$$

$$Z = (0.663)(10.01\%) + (0.337)(18.22\%) = 12.78\%$$

Table 12: Overall composite weight of the alternatives

	Factor A	Factor B	Composite Weight
(Adjusted) Weight	0.663	0.337	
Choice X	51.05%	11.49%	37.72%
Choice Y	38.93%	70.28%	49.49%
Choice Z	10.01%	18.22%	12.78%

For this example, we get the results that choice Y is the best choice, followed by X as the second choice and the worst choice is Z. The composite weights are ratio scale. We can say that choice Y is 3.87 times more preferable than choice Z, and choice Y is 1.3 times more preferable than choice X.

We can also check the overall consistency of hierarchy by summing for all levels, with weighted

consistency index (CI) in the nominator and weighted random consistency index (RI) in the denominator. Overall consistency of the hierarchy in our example above is given by

$$\overline{CR} = \frac{\sum_i w_i CI_i}{\sum_i w_i RI_i} = \frac{0.0897(1)+0.05(0.663)+0.04(0.337)}{0.90(1)+0.58(0.663)+0.58(0.337)} = 0.092 < 10\% \text{ (Acceptable)}$$

Frequently Asking Questions

How to dealing with high CR values?

Saaty suggested a threshold of 10% for the CR values. The author believes that the threshold of CR is analogue to significant level of statistical analysis that related to confidence level of the analysis rather than a fixed value that everybody must follow. If it is possible, the same respondents may be asked again to reconsider their answers. The original CR can be used as a feedback regarding their consistency.

Can we change the AHP scale to -9 to +9?

No, AHP matrix must be positive matrix. Thus, you need to rescale the negative value into positive value 1/9 to 9 with 1 means indifferent in judgment.

I have more than 20 criteria, why RI table is only until n=10?

As mentioned in the tutorial, the value of RI is provided by Prof. Saaty. It was generated using similar idea of Bootstrap using random reciprocal matrix of scale 1/9, 1/8, ... ,1, 2, .., 8, 9. In practice, you don't need RI for more than n=7 because the number of pair comparisons $n*(n-1)/2$ become so large. Too many questions on the same things make people confuse to answer them correctly. Utilize the hierarchy by grouping some criteria into a higher level hierarchy of the criteria.

How to aggregate AHP result for group decision?

For group decision, many researchers recommended to use geometric mean rather than arithmetic mean for aggregation between actors. Compute the priority matrix for each survey response first then make geometric average to aggregate the results.

Is it possible to obtain a negative consistency ratio in AHP?

Negative CR is theoretically not possible but it may happen due to error in the approximation of Eigen value. Remember that if you use MS Excel, it is only give you the approximation of Eigen value, not the real one. Please use Maxima (a computer algebra system that can be downloaded

from <http://maxima.sourceforge.net/>) to obtain the actual Eigen Values and Eigen Vectors. If you really want to use Eigen value and Eigen vector for real problem using MS excel, I suggest you to install free Add Ins library matrix.xla from <http://digilander.libero.it/foxes/SoftwareDownload.htm>. It is a great program that you may use many matrix functions.

Final Remark

By now you have learned several introductory methods on multi criteria decision making (MCDM) from simple cross tabulation, using rank, and weighted score until AHP. Using Analytic Hierarchy Process (AHP), you can convert ordinal scale to ratio scale and even check its consistency.